Decomposition of the vertex operator algebra $V_{\sqrt{2}A_3}$

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1 Introduction

A conformal vector with central charge c in a vertex operator algebra is an element of weight two whose component operators satisfy the Virasoro algebra relation with central charge c. Then the vertex operator subalgebra generated by the vector is isomorphic to a highest weight module for the Virasoro algebra with central charge c and highest weight 0 (cf. [M]).

Let $V_{\sqrt{2}A_l}$ be the vertex operator algebra associated with a lattice $\sqrt{2}A_l$, where $\sqrt{2}A_l$ denotes $\sqrt{2}$ times an ordinary root lattice of type A_l. Motivated by the problem of looking for maximal associative algebras of the Griess algebra [G], a class of conformal vectors in $V_{\sqrt{2}A_l}$ were studied and constructed in [DLMN]. It was shown in [DLMN] that the Virasoro element ω of $V_{\sqrt{2}A_l}$ is decomposed into a sum of $l+1$ mutually orthogonal conformal vectors ω^i; $1 \leq i \leq l+1$ with central charge $c_i = 1-6/(i+2)(i+3)$ for $1 \leq i \leq l$ and $c_{l+1} = 2l/(l+3)$. The vertex operator subalgebra generated by conformal vector ω^i is exactly the irreducible highest weight module $L(c_i, 0)$ for the Virasoro algebra. The vertex operator subalgebra $T = T_l$ generated by these conformal vectors is isomorphic to a tensor product $\otimes_{i=1}^{l+1} L(c_i, 0)$ of the Virasoro vertex operator algebras $L(c_i, 0)$ and $V_{\sqrt{2}A_l}$ is a direct sum of irreducible T-submodules.

In this paper we determine the decomposition of $V_{\sqrt{2}A_3}$ into the direct sum of irreducible T-modules completely. The direct summands have been determined [KMY] in the case $l = 2$. For general l only the direct summands with minimal weights at most two are known [Y].

The main idea for the decomposition in this paper is to embed $V_{\sqrt{2}A_3}$ into the vertex operator algebra $V_{(\sqrt{2}A_1)^{\oplus 3}}$ by considering the lattice $\sqrt{2}A_3$ as a sublattice of $(\sqrt{2}A_1)^{\oplus 3}$. It turns out that $V_{\sqrt{2}A_3}$ is isomorphic to the vertex operator subalgebra $V_{(\sqrt{2}A_1)^{\oplus 3}}^{+}$ which is the fixed points of the involution of $V_{(\sqrt{2}A_1)^{\oplus 3}}$ induced from the -1 isometry of $(\sqrt{2}A_1)^{\oplus 3}$. Moreover, $V_{(\sqrt{2}A_1)^{\oplus 3}}^{+}$ has a subalgebra isomorphic to $V_{\sqrt{2}A_2}^{+} \otimes V_F^{+}$ where F is a rank one lattice spanned by an element of square length 6 and $V_{(\sqrt{2}A_1)^{\oplus 3}}^{+}$ is a direct sum of 3 irreducible modules for $V_{\sqrt{2}A_2}^{+} \otimes V_F^{+}$. Then using [DG] on the decomposition of lattice type vertex operator algebra of rank 1 into the direct sum of irreducible modules for the Virasoro algebra and results in [KMY], we can determine all the irreducible T-modules in $V_{\sqrt{2}A_3}$. We should also mention that the sum of certain irreducible modules for T inside $V_{\sqrt{2}A_3}$ forms a rational vertex operator algebra from our picture.

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2 Some automorphisms of $V_{\mathbb{Z}\alpha}$

Our notation for the vertex operator algebra $V_L = M(1) \otimes \mathbb{C}[L]$ associated with a positive definite even lattice L is standard [FLM]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra and extend the the bilinear form to $\mathfrak{h}$ by $\mathbb{C}$-linearity, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ is the corresponding affine algebra, $M(1) = \mathbb{C}[\alpha(n)|\alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\mathfrak{h}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and n positive, and $K = 1$. The element in the group algebra $\mathbb{C}[L]$ of the additive group L corresponding to $\beta \in L$ will be denoted by e^{β}. Note that the central extension $\hat{L}$ of L by the cyclic group of order 2 is split if the square length of any element in L is a multiple of 4 (cf. [FLM]). For example, $\sqrt{2}A_l$ is a such lattice. The vacuum vector $\mathbf{1}$ of V_L is $1 \otimes e^0$ and the Virasoro element ω is $\frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)^2$ where $\{\beta_1, ..., \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$.

We need to know explicit expressions of the vertex operators $Y(u, z)$ for $u = h(-1)$ or $u = e^{\beta}$ for $h \in \mathfrak{h}$ and $\beta \in L$ in the next section to do certain calculations. We assume that the square length of any element in L is a multiple of 4. The operator $Y(h(-1), z)$ is defined as

$$Y(h(-1), z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} = \sum_{n \in \mathbb{Z}} h(-1)_n z^{-n-1} \tag{2.1}$$

where $h(n)$ acts on $M(1)$ if $n \neq 0$ and $h(0)$ acts on $\mathbb{C}[L]$ so that $h(0)e^{\gamma} = \langle h, \gamma \rangle e^{\gamma}$ for $\gamma \in L$. In order to define $Y(e^{\beta}, z)$ we need to define operators e_{β} and z^{β} acting on V_L such that $e_{\beta}(u \otimes e^{\gamma}) = u \otimes e^{\beta + \gamma}$ and $z^{\beta}(u \otimes e^{\gamma}) = z^{\langle \beta, \gamma \rangle} u \otimes e^{\gamma}$ for $u \in M(1)$ and $\gamma \in L$. Then

$$Y(e^{\beta}, z) = \sum_{n \in \mathbb{Z}} e_n^{\beta} z^{-n-1} = e^{\sum_{n<0} \frac{\beta(n)}{-n} z^{-n}} e^{\sum_{n>0} \frac{\beta(n)}{-n} z^{-n}} e_{\beta} z^{\beta}. \tag{2.2}$$

We refer the reader to [FLM] for the definition of vertex operators $Y(u, z)$ for general $u \in V_L$.

Let $L^\circ = \{\alpha \in \mathfrak{h} | \langle \alpha, L \rangle \subset \mathbb{Z}\}$ be the dual lattice of L. Then L° / L is a finite group. For each $\lambda \in L^\circ$ the corresponding untwisted Fock space $V_{L+\lambda} = M(1) \otimes \mathbb{C}[L + \lambda]$ is an irreducible module for V_L [FLM]. Let $L^\circ = \cup_{i \in L^\circ / L}(L + \lambda_i)$ be a coset decomposition. Then $V_{L+\lambda_i}$ are all inequivalent irreducible V_L-modules [D].

Let $V_{\mathbb{Z}\alpha}$ be the vertex operator algebra associated with a rank one lattice $\mathbb{Z}\alpha$ such that $\langle \alpha, \alpha \rangle = 2$. The homogeneous subspace $\mathfrak{g} = (V_{\mathbb{Z}\alpha})_{(1)}$ of $V_{\mathbb{Z}\alpha}$ of weight one possesses a Lie algebra structure given by $[u, v] = u_0 v$ with a symmetric invariant form $\langle \cdot, \cdot \rangle$ such that $\langle u, v \rangle \mathbf{1} = u_1 v$ ([FLM, Section 8.9]). We have

$$[e^\alpha, e^{-\alpha}] = \alpha(-1), \qquad\qquad [\alpha(-1), e^{\pm\alpha}] = \pm 2 e^{\pm\alpha},$$
$$\langle e^\alpha, e^{-\alpha} \rangle = 1, \qquad\qquad \langle \alpha(-1), \alpha(-1) \rangle = 2,$$

and $\langle u, v \rangle = 0$ for the other pairs u, v in $\{\alpha(-1), e^\alpha, e^{-\alpha}\}$. In particular, $\{\alpha(-1), e^\alpha, e^{-\alpha}\}$ is a standard basis of $\mathfrak{g} \cong sl_2(\mathbb{C})$.

Now consider three automorphisms (cf. [FLM]) θ_1, θ_2, σ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ of order two such that

$$\theta_1 : \alpha(-1) \longmapsto \alpha(-1), \qquad e^\alpha \longmapsto -e^\alpha, \qquad e^{-\alpha} \longmapsto -e^{-\alpha},$$
$$\theta_2 : \alpha(-1) \longmapsto -\alpha(-1), \qquad e^\alpha \longmapsto e^{-\alpha}, \qquad e^{-\alpha} \longmapsto e^\alpha,$$
$$\sigma : \alpha(-1) \longmapsto e^\alpha + e^{-\alpha}, \quad e^\alpha + e^{-\alpha} \longmapsto \alpha(-1), \quad e^\alpha - e^{-\alpha} \longmapsto -(e^\alpha - e^{-\alpha}).$$

Clearly $\sigma\theta_1\sigma = \theta_2$. These automorphisms of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ can be uniquely extended to automorphisms of the vertex operator algebra $V_{\mathbb{Z}\alpha}$. In order to see this we recall that the Verma module $V(1, 0)$ for the affine algebra $A_1^{(1)} = sl_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ is the quotient of $U(A_1^{(1)})$ modulo the left ideal generated by $x \otimes t^n$, $\mathbf{c} - 1$ for $x \in sl_2(\mathbb{C})$ and $n \geq 0$. Note that the automorphism group $\mathrm{Aut}(sl_2(\mathbb{C}))$ of the Lie algebra $sl_2(\mathbb{C})$ acts on $A_1^{(1)}$ by acting on the first tensor factor of $sl_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$ and trivially on $\mathbf{c}$. As a result $\mathrm{Aut}(sl_2(\mathbb{C}))$ acts on $U(A_1^{(1)})$ as algebra automorphisms. Clearly this induces an action of $\mathrm{Aut}(sl_2(\mathbb{C}))$ on $V(1, 0)$. Note that $V_{\mathbb{Z}\alpha}$ is the irreducible quotient of $V(1, 0)$ modulo the maximal submodule for $A_1^{(1)}$. It is easy to see from this construction that $\mathrm{Aut}(sl_2(\mathbb{C}))$ acts on $V_{\mathbb{Z}\alpha}$. In fact, the subgroup of $\mathrm{Aut}(sl_2(\mathbb{C}))$ consisting of those preserving the invariant bilinear form can be regarded as a subgroup of the automorphisms of the vertex operator algebra $V_{\mathbb{Z}\alpha}$. (This observation works for any finite dimensional semisimple Lie algebra in the position of $sl_2(\mathbb{C})$.) Since θ_1, θ_2 and σ preserve the bilinear form on $sl_2(\mathbb{C})$ they act on $V_{\mathbb{Z}\alpha}$ as vertex operator algebra automorphisms.

We denote the corresponding automorphisms of $V_{\mathbb{Z}\alpha}$ by the same symbols θ_1, θ_2, and σ. Then on $V_{\mathbb{Z}\alpha}$, we have $\theta_1(u \otimes e^\beta) = (-1)^{\langle \alpha, \beta \rangle / 2} u \otimes e^\beta$ for $u \in M(1)$ and $\beta \in \mathbb{Z}\alpha$ and θ_2 is the automorphism induced from the isometry $\beta \longmapsto -\beta$ of $\mathbb{Z}\alpha$ [FLM]. We still have $\sigma\theta_1\sigma = \theta_2$. We also have

$$\sigma(\alpha(-1)^2) = \alpha(-1)^2 \quad \text{and} \quad \sigma(e^{\pm\alpha}) = \frac{1}{2}(\alpha(-1) \mp (e^\alpha - e^{-\alpha})).$$

We should mention that $\sigma(\alpha(-1)^2) = \alpha(-1)^2$ is not obvious. Using the definitions of

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$Y(e^{\pm\alpha}, z)$ and σ we see that

$$\begin{aligned}
\sigma(\alpha(-1)^2) &= \sigma(\alpha(-1))_{-1}\sigma(\alpha(-1)) \\
&= (e^\alpha + e^{-\alpha})_{-1}(e^\alpha + e^{-\alpha}) \\
&= \alpha(-1)^2.
\end{aligned}$$

3   Decomposition of $V_{\sqrt{2}A_3}$

Let L be a lattice with basis $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $\langle\alpha_i, \alpha_j\rangle = 2\delta_{ij}$. Then $L = A_1 \oplus A_1 \oplus A_1$ where A_1 is the root lattice of $sl_2(\mathbb{C})$. Set

$$\beta_1 = (\alpha_1 + \alpha_2)/\sqrt{2}, \qquad \beta_2 = (-\alpha_2 + \alpha_3)/\sqrt{2}, \qquad \beta_3 = (-\alpha_1 + \alpha_2)/\sqrt{2}.$$

Then $\{\beta_1, \beta_2, \beta_3\}$ forms the set of simple roots of type A_3. Set $\gamma = -\alpha_1 + \alpha_2 + \alpha_3$. We consider two sublattices of L :

$$N = \sum_{i,j=1}^{3} \mathbb{Z}(\alpha_i \pm \alpha_j), \qquad D = E \oplus F,$$

where $E = \mathbb{Z}(\alpha_1 + \alpha_2) + \mathbb{Z}(-\alpha_2 + \alpha_3) = \mathbb{Z}\sqrt{2}\beta_1 + \mathbb{Z}\sqrt{2}\beta_2$ and $F = \mathbb{Z}\gamma$.

Lemma 3.1 (1) *We have that* $N = \{\beta \in L \,|\, \langle\alpha_1 + \alpha_2 + \alpha_3, \beta\rangle \equiv 0 \pmod 4\}$, N *is isometric to* $\sqrt{2}A_3$, *and* $[L : N] = 2$.

(2) $[L : D] = 3$ *and* $L = D \cup (D + \alpha_2) \cup (D - \alpha_2)$

(3) $\alpha_2 = \sqrt{2}(\beta_1 - \beta_2)/3 + \gamma/3$ *and each element of the coset* $D + \alpha_2$ *can be uniquely written as an orthogonal sum of an element in* $E + \sqrt{2}(\beta_1 - \beta_2)/3$ *and an element in* $F + \gamma/3$.

(4) E *is isometric to* $\sqrt{2}A_2$.

Proof (1) The first assertion can be verified easily. Since $N = \mathbb{Z}\sqrt{2}\beta_1 + \mathbb{Z}\sqrt{2}\beta_2 + \mathbb{Z}\sqrt{2}\beta_3$, the second assertion holds. We have $\alpha_i \notin N$ and $L = N \cup (N + \alpha_i)$ for any i. In particular $[L : N] = 2$.

(2)–(4) are obvious. $\square$

Since E and F are even lattices V_E and V_F are vertex operator algebras which can be regarded as vertex operator subalgebras of $V_{\sqrt{2}A_1^{\oplus 3}}$ (with different Virasoro algebras). Note that $\langle E + \sqrt{2}(\beta_1 - \beta_2)/3, E\rangle \subset \mathbb{Z}$ and $\langle F + \gamma/3, F\rangle \subset \mathbb{Z}$. Thus $V_{E+\sqrt{2}(\beta_1-\beta_2)/3}$ is an irreducible V_E-module and $V_{F+\gamma/3}$ is an irreducible V_F-module (cf. [FLM]).

The lattice L is a direct sum of $\mathbb{Z}\alpha_i$; $i = 1, 2, 3$ and thus the vertex operator algebra V_L associated with L is a tensor product $V_L = V_{\mathbb{Z}\alpha_1} \otimes V_{\mathbb{Z}\alpha_2} \otimes V_{\mathbb{Z}\alpha_3}$ (see [FHL] for the definition of tensor product vertex operator algebra). Define three automorphisms of V_L of order two by

$$\psi_1 = \theta_1 \otimes \theta_1 \otimes \theta_1, \qquad \psi_2 = \theta_2 \otimes \theta_2 \otimes \theta_2, \qquad \tau = \sigma \otimes \sigma \otimes \sigma,$$

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where θ_1, θ_2, and σ are the automorphisms of $V_{\mathbb{Z}\alpha_i}$ described in Section 2. Then

$$\psi_1(u \otimes e^\beta) = (-1)^{\langle \alpha_1 + \alpha_2 + \alpha_3, \beta \rangle / 2} u \otimes e^\beta$$

for $u \in M(1)$ and $\beta \in L$, ψ_2 is the automorphism induced from the isometry $\beta \longmapsto -\beta$ of L, and $\tau\psi_1\tau = \psi_2$.

For any ψ_2-invariant subspace U of V_L we shall write $U^{\pm} = \{v \in V_L \mid \psi_2(v) = \pm v\}$.

Lemma 3.2 *We have* $\tau(V_N) = V_L^+$.

Proof From the action of ψ_1 on V_L and Lemma 3.1 (1) it follows that $V_N = \{v \in V_L \mid \psi_1(v) = v\}$. Since $\psi_2\tau = \tau\psi_1$, the assertion holds. $\quad\square$

Lemma 3.3 (1) $V_L^+ \cong V_D^+ \oplus V_{D+\alpha_2}$ *as* V_D^+-*modules.*
(2) $V_D^+ = (V_E^+ \otimes V_F^+) \oplus (V_E^- \otimes V_F^-)$.
(3) $V_{D+\alpha_2} = V_{E+\sqrt{2}(\beta_1-\beta_2)/3} \otimes V_{F+\gamma/3}$.

Proof Lemma 3.1 (2) implies $V_L = V_D \oplus V_{D+\alpha_2} \oplus V_{D-\alpha_2}$. Since ψ_2 leaves V_D invariant and interchanges $V_{D+\alpha_2}$ and $V_{D-\alpha_2}$, we have

$$\begin{aligned} V_L^+ &= V_D^+ \oplus (V_{D+\alpha_2} \oplus V_{D-\alpha_2})^+ \\ &\cong V_D^+ \oplus V_{D+\alpha_2} \end{aligned}$$

as V_D^+-modules. Now $D = E \oplus F$ and both of E and F are ψ_2-invariant. Hence (2) holds. (3) follows from Lemma 3.1 (3). $\quad\square$

Similarly we have the following result which is not used in this paper to study the decomposition of V_N into the sum of irreducible T-modules.

Lemma 3.4 (1) $V_L^- \cong V_D^- \oplus V_{D+\alpha_2}$ *as* V_D^+-*modules.*
(2) $V_D^- = (V_E^+ \otimes V_F^-) \oplus (V_E^- \otimes V_F^+)$.

Let us recall the conformal vectors introduced in [DLMM]. For any positive integer $l > 0$ let Φ_l be the root system of type A_l generated by simple roots $\beta_1, ..., \beta_l$ and Φ_l^+ be the set of positive roots. We assume that the square length of a root is 2. We also assume that Φ_i is a sub-system of Φ_l with simple roots $\beta_1, ...\beta_i$ for $i \leq l$. Let $\sqrt{2}A_i$ be the positive definite even lattice spanned by $\sqrt{2}\Phi_i$. Then $\sqrt{2}A_i$ is a sublattice of $\sqrt{2}A_l$ and $V_{\sqrt{2}A_i}$ is a vertex operator subalgebra of $V_{\sqrt{2}A_l}$.

For $\beta \in \Phi_l$ set

$$w_\beta^{\pm} = \beta(-1)^2 \pm 2(e^{\sqrt{2}\beta} + e^{-\sqrt{2}\beta})$$

and

$$s^i = \frac{1}{2(i+3)} \sum_{\beta \in \Phi_i^+} w_\beta^-, \qquad \omega = \frac{1}{2(l+1)} \sum_{\beta \in \Phi_l^+} \beta(-1)^2.$$

Then the conformal vectors of $V_{\sqrt{2}A_l}$ defined in [DLMN] are

$$\omega^1 = s^1, \qquad \omega^{i+1} = s^{i+1} - s^i, i = 1, ..., l-1, \qquad \omega^{l+1} = \omega - s^l. \qquad (3.1)$$

Note that ω is the Virasoro element.

By Lemma 3.2, V_N is isomorphic to V_L^+ as a vertex operator algebra. We next calculate the images of the conformal vectors ω^1, ω^2, ω^3, and ω^4 in V_N under the automorphism τ. Note that $\Phi_2^+ = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$ and $\Phi_3^+ = \Phi_2^+ \cup \{\beta_3, \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3\}$.

Let $L(c, h)$ be the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. Then $L(c, 0)$ is a vertex operator algebra if $c \neq 0$ (cf. [FZ]). The conformal vectors ω^1, ω^2, ω^3, and ω^4 are mutually orthogonal and their central charges are $1/2$, $7/10$, $4/5$, and 1. Hence the subalgebra T of V_N generated by these conformal vectors is isomorphic to

$$L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 0) \otimes L(1, 0).$$

Moreover, V_N is a completely reducible T-module. The main purpose in this paper is to determine the irreducible T-modules in V_N or equivalently in V_L^+.

In order to achieve this we need to determine the images of s^i under τ.

Lemma 3.5 *We have*

$$\tau(s^1) = \frac{1}{8} w_{\beta_3}^+,$$

$$\tau(s^2) = \frac{1}{10}(w_{\beta_3}^+ + w_{\beta_2}^- + w_{\beta_2+\beta_3}^+),$$

$$\tau(s^3) = \frac{1}{12}(w_{\beta_3}^+ + w_{\beta_2}^- + w_{\beta_2+\beta_3}^+ + w_{\beta_3}^- + w_{\beta_2+\beta_3}^- + w_{\beta_2}^+)$$

$$= \frac{1}{6}(\beta_2(-1)^2 + \beta_3(-1)^2 + (\beta_2 + \beta_3)(-1)^2),$$

$$\tau(\omega) = \omega.$$

Proof The proof is a straightforward computation. We compute $\tau(s^1)$ here and leave the others to the reader. Note that

$$s^1 = \frac{1}{8}\left(\beta_1(-1)^2 - 2(e^{\sqrt{2}\beta_1} + e^{-\sqrt{2}\beta_1})\right)$$

$$= \frac{1}{16}(\alpha_1(-1) + \alpha_2(-1))^2 - \frac{1}{4}(e^{\alpha_1}e^{\alpha_2} + e^{-\alpha_1}e^{-\alpha_2}),$$

where $e^{\pm\alpha_1}e^{\pm\alpha_2}$ is understood as tensor product of $e^{\pm\alpha_1}$ with $e^{\pm\alpha_2}$ in $V_{\mathbb{Z}\alpha_1} \otimes V_{\mathbb{Z}\alpha_2}$. Recall the definitions of vertex operators $Y(h(-1), z)$ and $Y(e^\beta, z)$ from (2.1) and (2.2). Then

from the definition of τ we obtain

$$\tau(s^1) = \frac{1}{16}(e^{\alpha_1} + e^{-\alpha_1} + e^{\alpha_2} + e^{-\alpha_2})_{-1}(e^{\alpha_1} + e^{-\alpha_1} + e^{\alpha_2} + e^{-\alpha_2})$$

$$- \frac{1}{16}(\alpha_1(-1) - (e^{\alpha_1} - e^{-\alpha_1}))(\alpha_2(-1) - (e^{\alpha_2} - e^{-\alpha_2}))$$

$$- \frac{1}{16}(\alpha_1(-1) + (e^{\alpha_1} - e^{-\alpha_1}))(\alpha_2(-1) + (e^{\alpha_2} - e^{-\alpha_2}))$$

$$= \frac{1}{16}(\alpha_1(-1)^2 + \alpha_2(-1)^2) + \frac{1}{8}(e^{\alpha_1+\alpha_2} + e^{\alpha_1-\alpha_2} + e^{-\alpha_1+\alpha_2} + e^{-\alpha_1-\alpha_2})$$

$$- \frac{1}{8}\alpha_1(-1)\alpha_2(-1) - \frac{1}{8}(e^{\alpha_1+\alpha_2} - e^{\alpha_1-\alpha_2} - e^{-\alpha_1+\alpha_2} + e^{-\alpha_1-\alpha_2})$$

$$= \frac{1}{8}(\beta_3(-1)^2 + 2(e^{\sqrt{2}\beta_3} + e^{-\sqrt{2}\beta_3}))$$

$$= \frac{1}{8}w_{\beta_3}^+.$$

$\square$

Clearly, $\tau(s^1), \tau(s^2 - s^1), \tau(s^3 - s^2)$ are not the conformal vectors associated to the lattice $\sqrt{2}A_2$ defined in [DLMM]. We shall compose τ with another automorphism of V_L so that the resulting conformal vectors are those in [DLMM] and we can apply the decomposition result given in [KMY] for $V_{\sqrt{2}A_2}$.

Let φ be the automorphism of V_L defined by

$$\varphi : u \otimes e^\beta \longmapsto (-1)^{\langle -\alpha_2+\alpha_3, \beta\rangle/2} u \otimes e^\beta$$

for $u \in M(1)$ and $\beta \in L$ and set

$$\rho = (\theta_2 \otimes 1 \otimes 1)\varphi\tau,$$

where $\theta_2 \otimes 1 \otimes 1$ is the automorphism which acts as θ_2 on $V_{\mathbb{Z}\alpha_1}$ and acts as the identity on $V_{\mathbb{Z}\alpha_2} \otimes V_{\mathbb{Z}\alpha_3}$. Let $\widetilde{\omega}^i = \rho(\omega^i)$.

Lemma 3.6 (1) *We have*

$$\rho(s^1) = \frac{1}{8}w_{\beta_1}^-, \qquad\qquad \rho(s^2) = \frac{1}{10}\sum_{\beta\in\Phi_2^+} w_\beta^-,$$

$$\rho(s^3) = \frac{1}{6}(\beta_1(-1)^2 + \beta_2(-1)^2 + (\beta_1 + \beta_2)(-1)^2), \qquad \rho(\omega) = \omega.$$

(2) *The $\widetilde{\omega}^1$, $\widetilde{\omega}^2$, and $\widetilde{\omega}^3$ are the mutually orthogonal conformal vectors of $V_E \cong V_{\sqrt{2}A_2}$ defined in [DLMN] and*

$$\widetilde{\omega}^4 = \rho(\omega) - \rho(s^3) = \frac{1}{12}\gamma(-1)^2$$

is the Virasoro element of V_F with central charge 1.

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Proof (2) follows from (1) and the definition of the conformal vectors in $V_{\sqrt{2}A_2}$ given in (3.1).

(1) follows from Lemma 3.5 and the definitions of all automorphisms involved. For example,

$$
\begin{aligned}
\rho(s^1) &= \frac{1}{8}((\theta_2 \otimes 1 \otimes 1)\varphi)w_{\beta_3}^+ \\
&= \frac{1}{16}((\theta_2 \otimes 1 \otimes 1)\varphi)((\alpha_1(-1) - \alpha_2(-1))^2 + 4(e^{\alpha_1}e^{-\alpha_2} + e^{-\alpha_1}e^{\alpha_2})) \\
&= \frac{1}{16}(\theta_2 \otimes 1 \otimes 1)((\alpha_1(-1) - \alpha_2(-1))^2 - 4(e^{\alpha_1}e^{-\alpha_2} + e^{-\alpha_1}e^{\alpha_2})) \\
&= \frac{1}{16}((\alpha_1(-1) + \alpha_2(-1))^2 - 4(e^{-\alpha_1}e^{-\alpha_2} + e^{\alpha_1}e^{\alpha_2})) \\
&= \frac{1}{8}w_{\beta_1}^-.
\end{aligned}
$$

$\square$

Let $\widetilde{T}'$ be the subalgebra generated by $\widetilde{\omega}^1$, $\widetilde{\omega}^2$, and $\widetilde{\omega}^3$, and $\widetilde{T}''$ the subalgebra generated by $\widetilde{\omega}^4$. Then $\widetilde{T}' \subset V_E^+$ and $\widetilde{T}'' \subset V_F^+$. Moreover,

$$
\widetilde{T}' \cong L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 0), \qquad \widetilde{T}'' \cong L(1, 0).
$$

As a $\widetilde{T}'$-module $V_E^{\pm}$ decomposes into a direct sum of irreducible $\widetilde{T}'$-submodules of the form $L(\tfrac{1}{2}, h_1) \otimes L(\tfrac{7}{10}, h_2) \otimes L(\tfrac{4}{5}, h_3)$. It follows from [KMY, Lemma 4.1] that V_E^+ is a direct sum of four irreducible submodules, which are isomorphic to

$$
\begin{array}{ll}
L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 0), & L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{7}{5}), \\
L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{7}{5}), & L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, 0),
\end{array} \tag{3.2}
$$

and V_E^- is a direct sum of four irreducible submodules, which are isomorphic to

$$
\begin{array}{ll}
L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{2}{5}), & L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{2}{5}), \\
L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 3), & L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, 3).
\end{array} \tag{3.3}
$$

In [KMY, Lemma 4.2] the irreducible $\widetilde{T}'$-submodules with minimal weight 2/3 of $V_{E+\sqrt{2}(\beta_1-\beta_2)/2}$, which is denoted by V^2 in [KMY], are determined. By using fusion rules ([DMZ], [W]) we see that as a $\widetilde{T}'$-module $V_{E+\sqrt{2}(\beta_1-\beta_2)/2}$ is a direct sum of four irreducible submodules, which are isomorphic to

$$
\begin{array}{ll}
L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, \tfrac{2}{3}), & L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{1}{15}), \\
L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{1}{15}), & L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, \tfrac{2}{3}).
\end{array} \tag{3.4}
$$

8

The decompositions of $V_F^{\pm}$ and $V_{F+\gamma/3}$ as $\widetilde{T}''$-modules can be found in [DG]; that is,

$$V_F^+ \cong (\oplus_{m\geq 0}L(1,\, 4m^2)) \oplus (\oplus_{m\geq 1}L(1,\, 3m^2)),$$
$$V_F^- \cong (\oplus_{m\geq 0}L(1,\, (2m+1)^2)) \oplus (\oplus_{m\geq 1}L(1,\, 3m^2)), \qquad (3.5)$$
$$V_{F+\gamma/3} \cong \oplus_{m\in\mathbb{Z}}L(1,\, (3m+1)^2/3).$$

From these decompositions and Lemma 3.3 we know all irreducible direct summands of V_L^+ as a $\widetilde{T}' \otimes \widetilde{T}''$-module.

Finally, note that the automorphisms $(\theta_2 \otimes 1 \otimes 1)\varphi$ and ψ_2 of V_L commute. Thus $\rho\psi_1 = \psi_2\rho$ and $\rho(V_N) = V_L^+$. Since $\rho(T) = \widetilde{T}' \otimes \widetilde{T}''$, the decomposition of V_N as a T-module and the decomposition of V_L^+ as a $\widetilde{T}' \otimes \widetilde{T}''$-module are the same. Using (3.2), (3.3), (3.4), (3.5), and Lemma 3.3 we conclude:

Theorem 3.7 *The decomposition of V_N into a direct sum of irreducible T-submodules is as follows:*

$$
\begin{aligned}
V_{\sqrt{2}A_3} = &\left(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 0) \bigoplus L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{7}{5}) \right. \\
&\left. \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{7}{5}) \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, 0) \right) \\
&\bigotimes \left((\oplus_{m\geq 0}L(1,\, 4m^2)) \bigoplus (\oplus_{m\geq 1}L(1,\, 3m^2)) \right) \\
&\bigoplus \left(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{2}{5}) \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{2}{5}) \right. \\
&\left. \bigoplus L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 3) \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, 3) \right) \\
&\bigotimes \left((\oplus_{m\geq 0}L(1,\, (2m+1)^2)) \bigoplus (\oplus_{m\geq 1}L(1,\, 3m^2)) \right) \\
&\bigoplus \left(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, \tfrac{2}{3}) \bigoplus L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5}) \otimes L(\tfrac{4}{5}, \tfrac{1}{15}) \right. \\
&\left. \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\tfrac{4}{5}, \tfrac{1}{15}) \bigoplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(\tfrac{4}{5}, \tfrac{2}{3}) \right) \\
&\bigotimes \left(\oplus_{m\in\mathbb{Z}}L(1,\, (3m+1)^2/3) \right).
\end{aligned}
$$

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